STRONG STEIN NEIGHBORHOOD BASES

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ABSTRACT. Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. We give several characterizations for the closure of Ω to have a Stein neighborhood basis in the sense that Ω has a defining function ρ such that $\{z \in \mathbb{C}^n : \rho(z) < \varepsilon\}$ is pseudoconvex for sufficiently small $\varepsilon > 0$. We also show that this condition is invariant under proper holomorphic maps that extend smoothly up to the boundary.

1. INTRODUCTION

A domain $\Omega \subset \mathbb{C}^n$ is called a domain of holomorphy if there exists a holomorphic function on Ω that cannot be "extended" past any boundary point. Any domain in $\mathbb{C}$ is a domain of holomorphy. However, Hartogs in 1906 discovered that not every domain in $\mathbb{C}^n$ for $n \geq 2$ is a domain of holomorphy. This fundamental discovery led to the characterization of domains of holomorphy called the Levi problem. The Levi problem was first solved by Oka in 1930's for $n = 2$ and by Bremermann, Norguet, and Oka in 1950's for $n \geq 3$. The solution of the Levi problem revealed a very interesting fact about domains of holomorphy: domains of holomorphy are precisely the so-called pseudoconvex domains and hence can be exhausted by pseudoconvex subdomains. That is, one can "approximate" a domain of holomorphy (or a pseudoconvex domain) from inside by pseudoconvex domains. Therefore, it is natural to ask whether it is possible to approximate such domains from outside. We refer the reader to [Hör90, Kra01, Ran86] for precise definitions and basic facts about domains of holomorphy and pseudoconvex domains.

A compact set $K \Subset \mathbb{C}^n$ is said to have a Stein neighborhood basis if for any domain V containing K there exists a pseudoconvex domain Ω_V such that $K \subset \Omega_V \subset V$. It is worth noting that the closure of the Hartogs triangle, $\{(z, w) \in \mathbb{C}^2 : 0 \leq |z| < |w| < 1\}$, does not have a Stein neighborhood basis. However, the Hartogs triangle is not smooth. In 1977, Diederich and Fornæss([DF77a]) found a smooth bounded pseudoconvex domain, the so-called worm domain, whose closure does not have a Stein neighborhood basis, thus answering in the negative a question of Behnke

and Thullen([BT33]). Subsequently, the question of when a Stein neighborhood basis exists has been studied by Bedford and Fornæss [BF78], Diederich and Fornæss [DF77a, DF77b], Sibony [Sib87a, Sib91], Stensønes [Ste87], et al.

The existence of a special kind of a Stein neighborhood basis is known to be connected to global regularity of the $\overline{\partial}$-Neumann problem [Str01], approximation properties for holomorphic functions [Čir69, FN77], and uniform algebras [Ros61].

In this paper we will concentrate on smooth domains and "smooth" means C^{∞}-smooth. However, the reader will notice that some of the results are still true for domains with C^3-smooth boundary. We are interested in the following stronger notion of Stein neighborhood bases for smooth domains, as it is fairly general and has many applications. We say the closure $\overline{\Omega}$ of a smooth bounded pseudoconvex domain Ω has a *strong Stein neighborhood basis* if Ω has a defining function ρ (see the Section 2 for a definition) and there exists $\varepsilon_0 > 0$ such that $\{z \in \mathbb{C}^n : \rho(z) < \varepsilon\}$ is pseudoconvex for $0 \leq \varepsilon \leq \varepsilon_0$. In Theorem 1 we give several characterizations for the closure to have a strong Stein neighborhood basis. The precise statement of Theroem 1 requires some technical definitions, and so is postponed to Section 3. We note that all smooth bounded pseudoconvex domains whose closure known to have a Stein neighborhood basis satisfy this condition. Whether it is equivalent to having a Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain is still open.

The existence of strong Stein neighborhood bases implies the so-called uniform H-convexity. A compact set $K \subset \mathbb{C}^n$ is said to be uniformly H-convex if there exists a positive sequence $\{\varepsilon_j\}$ that converges to 0, $c > 1$, and a sequence of pseudoconvex domains Ω_j such that $K \subset \Omega_j$ and $\varepsilon_j \leq \text{dist}(K, \mathbb{C}^n \setminus \Omega_j) \leq c\varepsilon_j$ for $j = 1, 2, \dots$. Čirka([Čir69]) showed that uniform H-convexity implies a "Mergelyan-like" approximation property for holomorphic functions. There are three conditions that are known to imply the existence of a (strong) Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$: having a holomorphic vector field in a neighborhood of the weakly pseudoconvex points that is transversal to the boundary [FN77], property $(\widetilde{P})$ [Sib87a, Sib91], and for domains in $\mathbb{C}^2$ having a defining function that is plurisubharmonic on the boundary [FH].

The following example shows that having a Stein neighborhood basis for the closure of a domain is not a invariant under biholomorphism in general.

Example 1. Let $\Omega_1 = \{(z, w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$ be the Hartogs triangle, and $\Omega_2 = \{(z, w) \in \mathbb{C}^2 : 0 \leq |z| < 1, 0 < |w| < 1\}$ be the bidisc in $\mathbb{C}^2$. Let $F : \Omega_1 \to \Omega_2$ be a biholomorphism defined as follows: $F(z, w) = (z/w, w)$. One can check that $\overline{F(\Omega_1)} = \overline{\Omega_2}$. Therefore, although $\overline{F(\Omega_1)}$ has a Stein neighborhood basis, $\overline{\Omega_1}$ does not.

It is still open whether having a Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain is invariant under biholomorphisms in general. However, an easy corollary to our main result (Theorem 1), is that having a strong Stein neighborhood basis is invariant under biholomorphisms that extend up to the boundary. More precisely,

Corollary 1. *Let Ω_1 and Ω_2 be two smooth bounded pseudoconvex domains in $\mathbb{C}^n, n \geq 2$. Assume that there exist a proper holomorphic map $F : \Omega_1 \to \Omega_2$ that extend smoothly to $\overline{\Omega}_1$ and $\overline{\Omega}_2$ has a strong Stein neighborhood basis. Then $\overline{\Omega}_1$ has a strong Stein neighborhood basis.*

At this point we would like to mention two open questions: Is the assumption of smooth extendibility of the biholomorphism in the Corollary above needed, or is it automatic? Does having a Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain imply the existence of a strong Stein neighborhood basis for the closure of the domain?

This paper is organized as follows: in Section 2 we set the notation and give basic definitions. In Section 3 we state the main theorem, Theorem 1, and give an application to a potential theoretic property the so-called property $(\widetilde{P})$ (see Corollary 2). In Section 4 we give the proof of Corollary 1 and Theorem 1. In Section 5 we give the proof of Corollary 2.

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2. NOTATION AND DEFINITIONS

Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and r be a defining function for Ω. That is, r is a smooth function defined in a neighborhood U of $\overline{\Omega}$ such that it is negative on Ω, positive on $U \setminus \overline{\Omega}$, zero on the boundary $b\Omega$ of Ω, and the gradient ∇r of r does not vanish on $b\Omega$. We define the complex Hessian of r at z as follows:

$$\mathcal{L}_r(z; A, \overline{B}) = \sum_{j,k=1}^{n} \frac{\partial^2 r(z)}{\partial z_j \partial \bar{z}_k} a_j \bar{b}_k,$$

where $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ are vectors in $\mathbb{C}^n$. We would like to note that we identify $\mathbb{C}^n$ with the $(1,0)$ tangent bundle of $\mathbb{C}^n$. Namely, $(a_1, \ldots, a_n)$ is identified with $\sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}$. We will denote $\mathcal{L}_r(z; A, \overline{A})$ by $\mathcal{L}_r(z; A)$, and $\sum_{j=1}^{n} |w_j|^2$ by $\|W\|^2$, where $W = (w_1, \ldots, w_n) \in \mathbb{C}^n$. Let $\vec{n}(z) \in \mathbb{R}^{2n}$ be the unit outward normal vector of $b\Omega$ at z. We denote the directional derivative in the direction $\vec{n}(z)$ at the point z by $\frac{d}{d\vec{n}(z)}$.

To simplify the notation and ease the calculation we will use the following notation:

$$\frac{d}{dr(z)} = \frac{1}{\|\nabla r(z)\|}\frac{d}{d\vec{n}(z)} \quad \text{and} \quad A(h)(z) = \sum_{j=1}^{n}\frac{\partial h(z)}{\partial z_j}a_j$$

for a (type $(1,0)$) vector $A = (a_1,\ldots,a_n) \in \mathbb{C}^n$ and $h \in C^1(\overline{\Omega})$. It is a standard fact that a smooth bounded domain Ω is pseudoconvex if and only if $\mathcal{L}_r(z;W) \geq 0$ for $z \in b\Omega$ and $W(r)(z) = 0$ where r is a defining function for Ω. One can check that this condition is independent of the defining function r. When we refer to finite or infinite type of a point in $b\Omega$, we mean type in the sense of D'Angelo [D82]. Let Ω_∞ denote the set of infinite type points of $b\Omega$ and

$$\Gamma_\Omega = \{(z,W) \in b\Omega \times \mathbb{C}^n : z \in \Omega_\infty, W(r)(z) = \mathcal{L}_r(z;W) = 0, \|W\| = 1\}.$$

Γ_Ω is, in some sense, the unit sphere of the weakly pseudoconvex directions on the infinite type points. For a fixed vector $A \in \mathbb{C}^n$ and $z \in b\Omega$ we will denote

$$
\begin{aligned}
C_r(z;A) &= \left.\frac{d\mathcal{L}_r(t;A)}{dr(z)}\right|_{t=z}, \\
D_r(z;A) &= \mathcal{L}_r(z;A,\overline{N}_r), \text{ and} \\
E_r(z;A) &= C_r(z;A) - 2\mathrm{Re}\left(D_r(z;A)\overline{A}(\ln\|\nabla r\|)(z)\right),
\end{aligned}
$$

where r is a defining function for Ω and $N_r = \frac{4}{\|\nabla r(z)\|^2}\sum_{j=1}^{n}\frac{\partial r}{\partial\bar{z}_j}\frac{\partial}{\partial z_j}$. Notice that $C_r(z;A)$ is the directional derivative of the complex Hessian $\mathcal{L}_r(z;A)$ in the (real) normal direction at z for a fixed vector A. C_r alone does not guarantee that the level sets outside the domain will be pseudoconvex because it measures how the complex Hessian changes as one moves out of the domain for only fixed vectors. On the other hand, $E_r(z;A)$ gives a sufficient condition for the existence of a Stein neighborhood basis (see Theorem 1) because it takes into account how the complex tangent vectors change as one moves out of the domain. This change can be measured by $D_r(z;A)$. We note that $D_r(z;A)$ plays a very crucial role in the vector field approach of Boas and Straube for the global regularity of the $\overline{\partial}$-Neumann problem [BS91, BS93, BS99]. This might suggest that there are deeper relations between the global regularity of the $\overline{\partial}$-Neumann problem and the existence of a Stein neighborhood basis for the closure.

3. STATEMENT OF THE MAIN THEOREM

The following is our main theorem. It gives several characterizations of having a strong Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain. It will be used in the proof of Corollary 1.

Theorem 1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. The following conditions are equivalent:*

i) *there exist a neighborhood U of $\overline{\Omega}$, a defining function ρ for Ω in U, and $c > 0$ such that $\mathcal{L}_\rho(z; W) \geq c\rho(z)\|W\|^2$ for $z \in U \setminus \Omega$ and $W(\rho)(z) = 0$,*

ii) *there exists a defining function ρ for Ω and $\varepsilon_0 > 0$ such that $\{z \in \mathbb{C}^n : \rho(z) < \varepsilon\}$ is pseudoconvex for $0 \leq \varepsilon \leq \varepsilon_0$. That is, $\overline{\Omega}$ has a strong Stein neighborhood basis*

iii) *there exists a defining function ρ for Ω such that*

$$E_\rho(z; W) \geq 0 \ for \ (z, W) \in \Gamma_\Omega.$$

iv) *there exist $h \in C^\infty(\overline{\Omega})$ and a defining function r for Ω such that:*

(1) $$\mathcal{L}_h(z; W) \geq |W(h)(z)|^2 + 2\mathrm{Re}\Big(D_r(z; W)\overline{W}(h)(z)\Big) - E_r(z; W)$$

for $(z, W) \in \Gamma_\Omega$.

Now we will give the definition of a potential theoretic condition: property $(\widetilde{P})$. The following definition is from [McN02].

Definition 1. Let Ω be a bounded domain in $\mathbb{C}^n$. Then Ω satisfies property $(\widetilde{P})$ if there exists a sequence of plurisubharmonic functions $\{\phi_j\} \subset C^\infty(\overline{\Omega})$ such that

i) $|W(\phi_j)(z)|^2 \leq \mathcal{L}_{\phi_j}(z; W)$ for $z \in \overline{\Omega}$ and $W \in \mathbb{C}^n$,
ii) $\mathcal{L}_{\phi_j}(z; W) \geq j\|W\|^2$ for $z \in b\Omega$ and $W \in \mathbb{C}^n$.

We note that if Ω is a C^3-smooth bounded pseudoconvex domain that satisfies property $(\widetilde{P})$ then for any defining function r there exists $h \in C^2(\overline{\Omega})$ such that (1) is satisfied [Şah06]. Property $(\widetilde{P})$ is related to another potential theoretic property called property (P). The difference is that instead of i) in the above definition, property (P) requires the sequence $\{\phi_j\}$ to be uniformly bounded on $\overline{\Omega}$. By exponentiating and scaling one can easily show that property (P) implies property $(\widetilde{P})$. However, it is still open whether the converse is true. Although these properties were introduced for studying compactness of the $\overline{\partial}$-Neumann problem ([Cat84, McN02]) they naturally appear in the study of Stein neighborhood bases. We note that Harrington([Har06]) showed that property (P) implies existence of a Stein neighborhood basis for the closure when the domain is C^1-smooth bounded and pseudoconvex.

As a result of our method we get a characterization of property $(\widetilde{P})$ in terms of existence of strong Stein neighborhood bases in some sense.

Corollary 2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. Then Ω satisfies property $(\widetilde{P})$ if and only if for every $M > 0$ there exists a defining function ρ such that $E_\rho(z; W) > M$ for $(z, W) \in \Gamma_\Omega$.*

An immediate implication of the above corollary is that property $(\widetilde{P})$ can be localized onto weakly pseudoconvex directions on infinite type points. We note that the localization of property $(\widetilde{P})$ has been obtained by Çelik([Çe]) in his thesis before.

4. PROOF OF THEOREM 1 AND COROLLARY 1

The following Lemmas will be useful in the proof of Theorem 1.

Lemma 1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. Assume that Ω has a defining function r such that $E_r(z; W) \geq 0$ for $(z, W) \in \Gamma_\Omega$. Then there exists a defining function ρ for Ω such that $E_\rho(z; W) > 0$ for $(z, W) \in \Gamma_\Omega$.*

Proof. Without loss of generality we may assume that $\overline{\Omega}$ is contained in the ball centered at the origin and of radius $\tau = \operatorname{diam}(\Omega)$. Let us define $\rho(z) = r(z)e^{h(z)}$ where

$$h(z) = \frac{e^{\beta(|z|^2 - \tau^2)}}{2\beta\tau^2},$$

and $\beta = 4 \sup\{1 + |\mathcal{L}_r(z; N_r, \overline{W})|^2 : (z, W) \in \Gamma_\Omega\}$. Now we would like to calculate E_ρ in terms of r. We note that $\frac{d}{d\rho} = e^{-h}\frac{d}{dr}$ and $N_\rho = e^{-h}N_r$ on $b\Omega$. One can check that

$$D_\rho(z; W) = D_r(z; W) + W(h)(z)$$

$$\overline{W}(\ln \|\nabla\rho\|)(z) = \overline{W}(\ln \|\nabla r\|)(z) + \overline{W}(h)(z)$$

We note that

$$(2) \qquad \frac{dW(r)}{dr(z)}(z) = W(\ln \|\nabla r\|)(z)$$

because W is a fixed vector and

$$\frac{dW(r)}{dr(p)}(z)\bigg|_{z=p} = \frac{dW}{dr(p)}(r)(p) + \sum_{j=1}^n w_j \frac{\nabla r(p)}{\|\nabla r(p)\|^2} \cdot \nabla\left(\frac{\partial r}{\partial z_j}\right)(p)$$

$$= \frac{dW}{dr(p)}(r)(p) + \frac{1}{2}W\left(\ln \|\nabla r\|^2\right)(p)$$

We note that the second term in the first equality consists of a summation of dot product of vectors. Using (2) one can calculate that

$$C_\rho(z; W) = C_r(z; W) + \mathcal{L}_h(z; W) + |W(h)(z)|^2 + 2\operatorname{Re}\left(W(h)(z)\overline{W}(\ln \|\nabla r\|)(z)\right).$$

If we put the above calculations together we get

$$(3) \qquad E_\rho(z; W) = \mathcal{L}_h(z; W) - |W(h)(z)|^2 - 2\operatorname{Re}\left(D_r(z; W)\overline{W}(h)(z)\right) + E_r(z; W).$$

So we only need to show that

$$g(z, W) = \mathcal{L}_h(z; W) - |W(h)(z)|^2 - \sqrt{\beta}|W(h)(z)| > 0 \text{ for } (z, W) \in \Gamma_\Omega.$$

Let us denote $\sum_{j=1}^{n} w_j \bar{z}_j$ by $\langle W, z \rangle$. Then one can calculate that

$$|W(h)(z)| = \frac{e^{\beta(|z|^2 - \tau^2)}}{2\tau^2} |\langle W, z \rangle|, \text{ and}$$

$$\mathcal{L}_h(z; W) = \frac{e^{\beta(|z|^2 - \tau^2)}}{2\tau^2} \left(\|W\|^2 + \beta |\langle W, z \rangle|^2 \right).$$

Therefore, if we use the inequality $\sqrt{\beta} |\langle W, z \rangle| \leq \beta |\langle W, z \rangle|^2 + 1/4$, the assumption that $\overline{\Omega}$ is contained in the ball centered at the origin and of radius τ, and the fact that $\|W\| = 1$ we get

$$g(z, W) \geq \frac{e^{\beta(|z|^2 - \tau^2)}}{2\tau^2} \left(1 + \left(\beta - \frac{1}{2\tau^2} \right) |\langle W, z \rangle|^2 - \sqrt{\beta} |\langle W, z \rangle| \right)$$

$$\geq \frac{e^{\beta(|z|^2 - \tau^2)}}{2\tau^2} \left(\frac{3}{4} - \frac{\|z\|^2}{2\tau^2} \right).$$

Again since $\overline{\Omega}$ is contained in the ball centered at the origin and it is of radius τ we have

$$\frac{3}{4} - \frac{\|z\|^2}{2\tau^2} > 0 \text{ for } z \in \overline{\Omega}.$$

This completes the proof of Lemma 1. $\qquad \square$

Lemma 2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$ and K be a compact subset of $b\Omega$. Assume that z is of finite type for every $z \in K$ and $h \in C^\infty(\overline{\Omega})$ is given. Then for every $j > 0$ there exists $h_j \in C^\infty(\overline{\Omega})$ such that $|h_j - h| \leq 1/j$ uniformly on $\overline{\Omega}$ and $\mathcal{L}_{h_j}(z; W) \geq j\|W\|^2$ for $z \in K$ and $W \in \mathbb{C}^n$.*

Proof. Using Proposition 3 in [Sib87b] we can construct a smooth finite type pseudo-convex subdomain D such that $K \subset bD \cap b\Omega$. Similar construction is used in [Bell86]. Catlin([Cat84]) showed that finite type domains satisfy property (P). So D satisfies property (P). That is, there is a sequence of functions $f_j \in C^\infty(\overline{D})$ such that $1/2 \leq f_j \leq 3/4$ on $\overline{D}$ and $\mathcal{L}_{f_j}(z; W) \geq j^2\|W\|^2$ for $z \in bD$ and $W \in \mathbb{C}^n$. Let $\tilde{f}_j$ denote a smooth extension of f_j's to $\overline{\Omega}$ such that $0 \leq \tilde{f}_j \leq 1$ on $\overline{\Omega}$. We finish the proof by choosing $h_j = h + \frac{\tilde{f}_{j+k}}{j}$ for sufficiently large k. $\qquad \square$

Proof of Theorem 1. We note that i)$\Rightarrow$ ii) is trivial and using (3) iii) $\Rightarrow$ iv) is easy to see. To prove ii) implies iii) let us assume that Ω has a defining function ρ and there exists $\varepsilon_0 > 0$ such that $\{z \in \mathbb{C}^n : \rho(z) < \varepsilon\}$ is pseudoconvex for $0 \leq \varepsilon \leq \varepsilon_0$. Now we would like to differentiate $\mathcal{L}_\rho(z; W)$ in the outward normal direction for $(z, W) \in \Gamma_\Omega$. If we apply $\frac{d}{d\vec{n}(p)}$ to $\mathcal{L}_\rho(z; W)$ for any smooth vector field W of type $(1, 0)$ such that $W(\rho) = 0$ on a neighborhood of Ω_∞ and $(p, W(p)) \in \Gamma_\Omega$ (calculations are similar to the ones in

the proof of Lemma 1) we get

$$\frac{d\mathcal{L}_\rho(z;W)}{d\rho(p)}\bigg|_{z=p} = C_\rho(p;W(p)) - 2\mathrm{Re}\left(D_\rho(p;W)\overline{W}(\ln\|\nabla\rho\|)(p)\right) = E_\rho(p;W(p)).$$

So ii) implies that the right hand side of the above equality is nonnegative. Therefore, $E_\rho(p;W(p)) \geq 0$ for $(p, W(p)) \in \Gamma_\Omega$.

Let us prove that iv)$\Rightarrow$ i): We divide the proof into two parts. In the first part, we will produce a defining function whose sublevel sets are pseudoconvex (from Ω's side) outside of Ω in a neighborhood of the set of infinite type points. In the second part, using property (P), we will modify this defining function away from infinite type points to get a strong Stein neighborhood basis for the closure.

Analysis on Infinite Type Points. Using Lemma 1 we can assume that Ω has a defining function r and there exists a function $h \in C^\infty(\overline{\Omega})$ such that

$$\mathcal{L}_h(z;W) > |W(h)(z)|^2 + 2\mathrm{Re}\left(D_r(z;W)\overline{W}(h)(z)\right) - E_r(z;W)$$

for $(z, W) \in \Gamma_\Omega$. We extend h to $\mathbb{C}^n$ as a smooth function and call the extension h. We scale h, if necessary, so that there is a neighborhood of $\overline{\Omega}$ on which the conditions of the theorem are still satisfied. We define $\rho(z) = r(z)e^{h(z)}$. We will show that there exists a neighborhood V of Ω_∞, the set of infinite type points in $b\Omega$, such that $\nabla\rho$ is nonvanishing on V, and the complex Hessian of ρ is nonnegative on vectors complex tangential to the level sets of ρ in $V \setminus \Omega$. Since Ω is bounded and $\|\nabla\rho\|$ is continuous and strictly positive on $b\Omega$, the first part of the above argument follows immediately. It suffices to argue near a boundary point q because Ω is bounded.

Let $z \in \mathbb{C}^n \setminus \Omega$, and $W \in \mathbb{C}^n$, be a complex tangential vector to the level set of ρ at z. Namely,

$$(4) \qquad W(\rho)(z) = e^{h(z)}\left(W(r)(z) + r(z)W(h)(z)\right) = 0.$$

Now we will calculate the complex Hessian of ρ at z in the direction W. So first we differentiate ρ with respect to z_j to get

$$(5) \qquad \frac{\partial\rho}{\partial z_j}(z) = e^{h(z)}\frac{\partial r}{\partial z_j}(z) + r(z)e^{h(z)}\frac{\partial h}{\partial z_j}(z)$$

and if we differentiate (5) with respect to $\bar{z}_k$ we get

$$\begin{aligned}
\frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}(z) &= e^{h(z)}\frac{\partial^2 r}{\partial\bar{z}_k\partial z_j}(z) + e^{h(z)}\frac{\partial h}{\partial\bar{z}_k}(z)\frac{\partial r}{\partial z_j}(z) + e^{h(z)}\frac{\partial r}{\partial\bar{z}_k}(z)\frac{\partial h}{\partial z_j}(z) \\
&+ r(z)e^{h(z)}\frac{\partial h}{\partial\bar{z}_k}(z)\frac{\partial h}{\partial z_j}(z) + r(z)e^{h(z)}\frac{\partial^2 h}{\partial\bar{z}_k\partial z_j}(z).
\end{aligned}$$

Using (4) in the last equality we get

$$\mathcal{L}_\rho(z; W) = e^{h(z)} \left(r(z)\mathcal{L}_h(z; W) + \mathcal{L}_r(z; W) - r(z)|W(h)(z)|^2 \right).$$

We would like to show that there exists a neighborhood V of infinite type points Ω_∞ such that

(6) $$f(z, W) = r(z)\mathcal{L}_h(z; W) + \mathcal{L}_r(z; W) - r(z)|W(h)(z)|^2 \geq 0$$

for $z \in V \setminus \Omega$ and $W \in \mathbb{C}^n$ such that $W(\rho)(z) = 0$.

Claim: To prove (6) it is sufficient to prove that for any $p \in \Omega_\infty$ we have

(7) $$\frac{df(z, W(z))}{d\vec{n}(p)}\bigg|_{z=p} > 0$$

for any smooth vector field W of type $(1,0)$ such that $W(\rho) = 0$ on a neighborhood of Ω_∞ and $(p, W(p)) \in \Gamma_\Omega$.

Proof of Claim: Let us fix $q \in \Omega_\infty$. Using translation and rotation we can move q to the origin such that the y_n-axis is the outward normal direction at 0. There exists a neighborhood $\widetilde{U}$ of 0 on which $\frac{\partial \rho}{\partial z_n}$ does not vanish. If $W = (w_1, \ldots, w_n)$ is a complex tangential vector to the level set of ρ at $z \in \widetilde{U}$ (i.e. $W(\rho)(z) = 0$) then

(8) $$w_n = -\left(\left(\frac{\partial \rho(z)}{\partial z_n}\right)^{-1} \right) \sum_{j=1}^{n-1} \frac{\partial \rho(z)}{\partial z_j} w_j.$$

We introduce an auxiliary real valued function g as $g(z, W') = f(z, W)$, where $W = (w_1, \ldots, w_n)$ and $W' = (w_1, \ldots, w_{n-1})$, with w_n given by (8). We choose an open neighborhood U of 0 such that $U \Subset \widetilde{U}$. Let $S = \{W' \in \mathbb{C}^{n-1} : \|W'\| = 1\}$. Notice that $(\overline{U \cap b\Omega}) \times S$ is compact and it is enough to show that for every $(p, W'_p) \in (\overline{U \cap b\Omega}) \times S$ there exists a neighborhood U_p of (p, W'_p) in $(\widetilde{U} \cap b\Omega) \times S$ such that $g(z, W') \geq 0$ for $(z, W') \in (U_p \setminus \Omega) \times S$ and $W' \in \mathbb{C}^{n-1}$. Due to the continuity of the complex Hessian this is true for strongly pseudoconvex directions. However, (7) implies that this is also true for weakly pseudoconvex directions. So the proof of the claim is complete.

Let us differentiate $f(z, W(z))$ with respect to $r(z)$ at $p \in \Omega_\infty$. Using (6) we get:

(9) $$\mathcal{L}_h(p; W) + C_r(p; W) + 2Re\left(\mathcal{L}_r(p; W, d\overline{W}/dr) \right) - |W(h)(p)|^2.$$

Since W is a weakly pseudoconvex direction we only need to compute the complex normal component of $\frac{dW}{dr(p)}$ at p to estimate the third term of the above expression. Hence, we need to compute $\frac{dW}{dr(p)}(r)(p)$ which represents the following: first we differentiate W by $\frac{d}{dr(p)}$ at p then apply the result to r and evaluate at p. Now we use the same calculations used to derive (2) and differentiate the left hand side of $W(r)(z) + r(z)W(h)(z) =$

0 to get

$$\frac{d\{W(r)(z) + r(z)W(h)(z)\}}{dr(p)}\bigg|_{z=p} = \frac{dW}{dr(p)}(r)(p) + W(\ln\|\nabla r\|)(p) + W(h)(p).$$

Thus we have:

(10) $$\frac{dW}{dr(p)}(r)(p) + W(\ln\|\nabla r\|)(p) + W(h)(p) = 0.$$

If $Y = \tau N_r + \xi$ where ξ is the complex tangential component of Y then

$$\tau = \frac{Y(r)(p)}{N_r(r)(p)} = Y(r)(p).$$

Then using the above observation with (10) we conclude that the third term in (9) is equal to

$$-2\text{Re}\left(D_r(p; W)\left(\overline{W}(\ln\|\nabla r\|)(p) + \overline{W}(h)(p)\right)\right).$$

Hence by (3) we have $E_\rho(p; W) > 0$ for $(p, W(p)) \in \Gamma_\Omega$.

Modification Away From Infinite Type Points. In the first part of the proof we showed that there exist a defining function ρ for Ω such that $E_\rho(z; W) > 0$ for $(z, W) \in \Gamma_\Omega$. That is, there is a neighborhood V of Ω_∞ such that the level sets of ρ are strongly pseudoconvex (from Ω's side) in $V \setminus \overline{\Omega}$. Now we will modify ρ away from infinite type points to get a smooth defining function r that will satisfy i). Let $\rho_\lambda(z) = e^{\lambda\rho(z)} - 1$. One can show that

(11) $$E_{\rho_\lambda}(z; W) = \lambda\mathcal{L}_\rho(z; W) + E_\rho(z; W).$$

Then we can choose open sets V_1, V_2, V_3, and $\lambda > 1$ so that $\Omega_\infty \Subset V_1 \Subset V_2 \Subset V_3 \Subset V$ and $E_{\rho_\lambda}(z; W) > 0$ for $z \in V \cap b\Omega$ and $W(\rho)(z) = 0$. Let χ be a smooth increasing convex function on the real line so that $\chi(t) \equiv 0$ for $t \leq 0$ and $\chi(t) > 0$ for $t > 0$. Let us choose $A = \sup\{2 + \chi'(t) : 0 \leq t \leq 2\}$. Using Lemma 2 we can choose a sequence of functions $\phi_j \in C^\infty(\overline{\Omega})$ such that:

1) $-6\ln A < \phi_j < -\ln A$ on $\overline{\Omega}$,
2) $-6\ln A < \phi_j < -5\ln A$ on $V_1 \cap b\Omega$,
3) $-6\ln A < \phi_j < -3\ln A$ on $V_2 \cap b\Omega$,
4) $-2\ln A < \phi_j < -\ln A$ on $b\Omega \setminus V_3$, and
5) $\mathcal{L}_{\phi_j}(z; W) > jA^6\|W\|^2$ for $z \in b\Omega \setminus V_1$ and $W \in \mathbb{C}^n$.

Let $h_j = e^{\phi_j} - 1/2$. Then one can check that $\mathcal{L}_{h_j}(z; W) > A|W(h_j)(z)|^2 + j$ for $z \in b\Omega \setminus V_1$ and $W \in \mathbb{C}^n$. Let us choose $a = \frac{1}{A^3} - \frac{1}{2}$, $\chi_a(t) = \chi(t - a)$, and $\psi_j(z) = \chi_a \circ h_j(z)$.

Then $\psi_j \equiv 0$ in a neighborhood of $\overline{V_2 \cap b\Omega}$. One can calculate that

$$
\begin{aligned}
\mathcal{L}_{\psi_j}(z; W) &= \chi_a'(h_j(z))\mathcal{L}_{h_j}(z; W) + \chi_a''(h_j(z))|W(h_j)(z)|^2, \\
|W(\psi_j)(z)|^2 &= |\chi_a'(h_j(z))|^2|W(h_j)(z)|^2.
\end{aligned}
$$

Let $r(z) = \rho_\lambda(z)e^{\psi_j(z)}$. As in (3) one can calculate that

$$
E_r(z; W) = \mathcal{L}_{\psi_j}(z; W) - |W(\psi_j)(z)|^2 - 2\mathrm{Re}\left(D_{\rho_\lambda}(z; W)\overline{W}(\psi_j)(z)\right) + E_{\rho_\lambda}(z; W).
$$

Since $r(z) = \rho_\lambda(z)$ in a neighborhood of $\overline{V_2 \cap b\Omega}$ we only need to show that

$$
(12) \qquad \mathcal{L}_{h_j}(z; W) > (\chi_a'(h_j(z)) + 1)|W(h_j(z))|^2 + |D_{\rho_\lambda}(z; W)|^2 - \frac{E_{\rho_\lambda}(z; W)}{\chi_a'(h_j)}
$$

for $z \in b\Omega \setminus V_2$ and $W(r)(z) = 0$. Let us choose

$$
(13) \qquad j > \sup\left\{|D_{\rho_\lambda}(z; W)|^2 - \frac{E_{\rho_\lambda}(z; W)}{\chi_a'(h_k(z))} : z \in b\Omega, W(r)(z) = 0, k = 1, 2, \ldots\right\}.
$$

We note that χ_a and ϕ_j's are chosen so that $\chi_a' \geq 0$ and $E_{\rho_\lambda}(z; W) > 0$ for $z \in V \cap b\Omega$ and $W(\rho)(z) = 0$. E_{ρ_λ} can be negative outside V in some directions but there exists $b > 0$ such that $\chi_a'(h_j(z)) > b$ for $j = 1, 2, \ldots$ and $z \in b\Omega \setminus V_3$. So the right hand side of (13) is finite and, since $A = \sup\{2 + \chi'(t) : 0 \leq t \leq 2\}$, one can choose j so that (12) is satisfied. Hence we showed that $E_r(z; W) > 0$ for $z \in b\Omega$ and $W(r)(z) = 0$. Similar argument used in the proof of ii)$\Rightarrow$iii) shows that there exists a neighborhood U of $\overline{\Omega}$ and $c > 0$ such that $\mathcal{L}_r(z; W) \geq cr(z)\|W\|^2$ for $z \in U \setminus \Omega$ and $W(r)(z) = 0$. $\qquad\square$

Now we will give the proof of Corollary 1.

Proof of Corollary 1. Let $n_1(p)$ denote the unit outward normal of Ω_1 at a boundary point $p \in b\Omega_1$. Furthermore, let $q = F(p)$ and $n_2 = F(n_1)$. We note that n_2 is transversal to $b\Omega_2$ at q and i) in Theorem 1 implies that there exist a defining function ρ_2 for Ω_2 and $c_2 > 0$ such that for

$$
\frac{d}{dn_2(q)}\mathcal{L}_{\rho_2}(z; W(z))\bigg|_{z=q} > c_2\|W\|^2
$$

for $q \in b\Omega_2$ and $W(\rho_2) = 0$ in a neighborhood of $\overline{\Omega}_2$. Namely,

$$
(14) \qquad \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}_{\rho_2}(q - \varepsilon n_2(q); W(q - \varepsilon n_2(q)))}{\varepsilon} > c_2
$$

for $(q, W(q)) \in \Gamma_{\Omega_2}$. Let $\rho_1(z) = \rho_2(F(z))$ and extend F smoothly to some neighborhood of $\overline{\Omega}_1$. So ρ_1 is a defining function for Ω_1. Since F is proper it transforms the complex Hessian of ρ_2 to the complex Hessian of ρ_1. More precisely, let J_F denote the

complex Jacobian of F. That is,

$$J_F = \left\{ \frac{\partial F_j}{\partial z_k} \right\}_{j,k}.$$

Then one can show that $\mathcal{L}_{\rho_1}(z; W(z)) = \mathcal{L}_{\rho_2}(F(z), J_F W(F(z)))$ for $z \in \overline{\Omega}_1$ and $W(\rho_1)(z) = 0$ if an only if $J_F W(\rho_2)(F(z)) = 0$. Let us fix a smooth vector field $\widetilde{W}$ of type $(1,0)$ that is complex tangential to level sets of ρ_1 on $\overline{\Omega}_1$ and denote $W = J_F \widetilde{W}$. Since F is holomorphic and extends to the boundary (14) implies that

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{L}_{\rho_1}(p - \varepsilon n_1(p); \widetilde{W}(p - \varepsilon n_1(p)))}{\varepsilon} \geq 0$$

for $(p, \widetilde{W}(p)) \in \Gamma_{\Omega_1}$. So we have $E_{\rho_1}(z; A) \geq 0$ for $(z, A) \in \Gamma_{\Omega_1}$. Therefore, Theorem 1 implies that $\overline{\Omega}_1$ has a strong Stein neighborhood basis. □

5. PROOF OF COROLLARY 2

Lemma 3. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that for every $M > 0$ there exist a neighborhood U of $b\Omega$ and $\psi \in C^\infty(U)$ such that $|W(\psi)(z)|^2 \leq \mathcal{L}_\psi(z; W)$ and $\mathcal{L}_\psi(z; W) \geq M\|W\|^2$, for $z \in U$ and $W \in \mathbb{C}^n$. Then Ω satisfies property $(\widetilde{P})$.*

Proof. One can check that $\mathcal{L}_h(z; W) \geq |W(h)(z)|^2$ if and only if $-e^{-h}$ is plurisubharmonic at z in the direction W. Without loss of generality we may assume that $\psi \leq -1$ on U. Let $A = -\sup_{z \in b\Omega} e^{-\psi(z)}$. Since $\Omega \setminus U$ is compact we can use a theorem of Demailly [De87] to choose Green's functions $G_1, \cdots, G_k$ so that $G_1 + \cdots + G_k - e^{-\psi} < A - 1$ on $\Omega \setminus U$. Then $F(z) = \max\{G_1(z) + \cdots + G_k(z) - e^{-\psi(z)}, A - 1/2\}$ is a continuous plurisubharmonic function that can be extended to a neighborhood of $\overline{\Omega}$. Using convolution with an approximate identity we may choose a function $\widetilde{F} \in C^\infty(\overline{\Omega})$ that is plurisubharmonic on Ω and arbitrarily close to F uniformly on $\overline{\Omega}$. Let $f(z) = -\log(-F(z))$ and $\tilde{f}(z) = -\log(-\widetilde{F}(z))$. Since $\widetilde{F}$ is plurisubharmonic we have $\mathcal{L}_{\tilde{f}}(z; W) \geq |W(\tilde{f})(z)|^2$ for $z \in \overline{\Omega}$ and $W \in \mathbb{C}^n$. Now we need to show that the complex Hessian of $\tilde{f}$ is large enough on $b\Omega$. Since on a sufficiently small band close to $b\Omega$ we have $F(z) = G_1(z) + \cdots + G_k(z) - e^{-\psi(z)}$ one can show that

$$\mathcal{L}_f(z; W) = \frac{\mathcal{L}_g(z; W) + e^{-\psi(z)}(\mathcal{L}_\psi(z; W) - |W(\psi)(z)|^2)}{e^{-\psi(z)} - g(z)}$$
$$+ \frac{|W(g)(z) + e^{-\psi(z)}W(\psi)(z)|^2}{(e^{-\psi(z)} - g(z))^2}$$

where $g(z) = G_1(z) + \cdots + G_k(z)$. Since $g \equiv 0$ on $b\Omega$ we have $\mathcal{L}_f(z; W) \geq \mathcal{L}_\psi(z; W) - |W(\psi)(z)|^2$ on $b\Omega$. But we could have chosen ψ so that $\mathcal{L}_\psi(z; W) \geq 2|W(\psi)(z)|^2$ which would imply that $\mathcal{L}_f(z; W) \geq \frac{1}{2}\mathcal{L}_\psi(z; W)$ on $b\Omega$. We can choose $\tilde{f}$ sufficiently close to

f so that $\mathcal{L}_{\tilde{f}}(z; W) \geq \frac{1}{2}\mathcal{L}_{\psi}(z; W) - \|W\|^2$ for $z \in b\Omega$ and $W \in \mathbb{C}^n$. Hence, Ω satisfies property $(\widetilde{P})$. $\square$

Proof of Corollary 2. Using the proof of Theorem 1 one can easily prove that if Ω satisfies property $(\widetilde{P})$ then for every $M > 0$ there exists a defining function ρ such that $E_\rho(z; W) > M$ for $(z; W) \in \Gamma_\Omega$. To prove the other direction let us assume that for $M > 0$ there exists a defining function ρ such that $E_\rho(z; W) > M$ for $(z, W) \in \Gamma_\Omega$. Let us define $\rho_\lambda(z) = e^{\lambda\rho(z)} - 1$ and fix a defining function r for Ω. Then by (11) we can choose a large enough $\lambda > 1$ so that $E_{\rho_\lambda}(z; W) > M\|W\|^2$ for $z \in b\Omega$ and $W(\rho)(z) = 0$. Then there exist $h \in C^\infty(\overline{\Omega})$ such that $\rho_\lambda(z) = r(z)e^{h(z)}$. By (3) the condition $E_{\rho_\lambda}(z; W) > M$ implies that

$$
\begin{aligned}
\mathcal{L}_h(z; W) \;>\;& M\|W\|^2 + |W(h)(z)|^2 + 2\text{Re}\left(D_r(z; W)\overline{W}(h)(z)\right) - E_r(z; W) \\[2mm]
>\;& M\|W\|^2 + \frac{|W(h)(z)|^2}{2} - 2|D_r(z; W)|^2 - |E_r(z; W)|
\end{aligned}
$$

for $z \in b\Omega$ and $W \in \mathbb{C}^n$. Let $\tilde{h}(z) = h(z)/2$ and

$$
\widetilde{M} = \frac{M}{2} - \sup\left\{|D_r(z; W)|^2 + \frac{1}{2}|E_r(z; W)| : (z, W) \in b\Omega \times \mathbb{C}^n, \|W\| \leq 1\right\}.
$$

Since we can choose M as large as we wish for every $\widetilde{M} > 0$ there exist a neighborhood U of $b\Omega$ and $\tilde{h} \in C^\infty(U)$ such that $\mathcal{L}_{\tilde{h}}(z; W) > \widetilde{M} + |W(\tilde{h})|^2$ for $z \in U$ and $W \in \mathbb{C}^n$. Then Lemma 3 implies that Ω satisfies property $(\widetilde{P})$. $\square$

REFERENCES

[BF78] Eric Bedford and John Erik Fornæss, *Domains with pseudoconvex neighborhood systems*, Invent. Math. **47** (1978), no. 1, 1–27, available at http://gdz.sub.uni-goettingen.de.

[Bell86] Steve Bell, *Differentiability of the Bergman kernel and pseudolocal estimates*, Math. Z. **192** (1986), no, 3 467–472, available at http://gdz.sub.uni-goettingen.de.

[BS91] Harold P. Boas and Emil J. Straube, *Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary*, Math. Z. **206** (1991), no. 1, 81–88, available at http://gdz.sub.uni-goettingen.de.

[BS93] _____, *de Rham cohomology of manifolds containing the points of infinite type, and Sobolev estimates for the $\overline{\partial}$-Neumann problem*, J. Geom. Anal. **3** (1993), no. 3, 225–235.

[BS99] _____, *Global regularity of the $\overline{\partial}$-Neumann problem: a survey of the L^2-Sobolev theory*, Several complex variables (Berkeley, CA, 1995–1996), Math. Sci. Res. Inst. Publ., vol. 37, Cambridge Univ. Press, Cambridge, 1999, pp. 79–111, arXiv:math.CV/9612204.

[BT33] H. Behnke and P. Thullen, *Zur Theorie der Singularitäten der Funktionen mehrerer komplexen Veränderlichen. Das Konvergenzproblem der Regularitätshüllen*, Math. Ann. **108** (1933), 91–104, available at http://gdz.sub.uni-goettingen.de.

[Cat84] David W. Catlin, *Global regularity of the $\overline{\partial}$-Neumann problem*, Complex analysis of several variables (Madison, Wis., 1982), Proc. Sympos. Pure Math., vol. 41, Amer. Math. Soc., Providence, RI, 1984, pp. 39–49.

[Çe] Mehmet Çelik, Ph.D. thesis, Texas A&M University, TX, in preperation.

[Čir69]   E. M. Čirka, *Approximation by holomorphic functions on smooth manifolds in* **C**^{n}, Mat. Sb. (N.S.) **78 (120)** (1969), 101–123, Math. USSR Sb., **7** (1969), 95–113.

[D82]   John P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. (2) **115** (1982), no. 3, 615–637.

[De87]   Jean-Pierre Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. **194** (1987), no. 4, 519–564.

[DF77a]   Klas Diederich and John Erik Fornæss, *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. **225** (1977), no. 3, 275–292, available at `http://gdz.sub.uni-goettingen.de`.

[DF77b]   _____, *Pseudoconvex domains: existence of Stein neighborhoods*, Duke Math. J. **44** (1977), no. 3, 641–662.

[FH]   John Erik Fornaess and Anne-Katrin Herbig, *A note on plurisubharmonic defining functions in* **C**^{2}, arXiv:math.CV/0609105.

[FN77]   John Erik Fornæss and Alexander Nagel, *The Mergelyan property for weakly pseudoconvex domains*, Manuscripta Math. **22** (1977), no. 2, 199–208, available at `http://gdz.sub.uni-goettingen.de`.

[Har06]   Phillip S. Harrington, *Property (P) and Stein neighborhood bases on C^1 domains*, preprint.

[Hör90]   Lars Hörmander, *An introduction to complex analysis in several variables*, third ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990.

[Kra01]   Steven G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1992 edition.

[McN02]   Jeffery D. McNeal, *A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator*, J. Funct. Anal. **195** (2002), no. 1, 190–205.

[Ran86]   R. Michael Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986.

[Ros61]   Hugo Rossi, *Holomorphically convex sets in several complex variables*, Ann. of Math. (2) **74** (1961), 470–493.

[Sib87a] Nessim Sibony, *Une classe de domaines pseudoconvexes*, Duke Math. J. **55** (1987), no. 2, 299–319.

[Sib87b] _____, *Hypoellipticité pour l'opérateur $\overline{\partial}$*, Math. Ann. **276** (1987), no. 2, 279–290, available at `http://gdz.sub.uni-goettingen.de`.

[Sib91]   _____, *Some aspects of weakly pseudoconvex domains*, Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 199–231.

[Ste87]   Berit Stensønes, *Stein neighborhoods*, Math. Z. **195** (1987), no. 3, 433–436, available at `http://gdz.sub.uni-goettingen.de`.

[Str01]   Emil J. Straube, *Good Stein neighborhood bases and regularity of the $\overline{\partial}$-Neumann problem*, Illinois J. Math. **45** (2001), no. 3, 865–871, arXiv: math.CV/0006136.

[Şah06]   Sönmez Şahutoğlu, *Compactness of the $\overline{\partial}$-Neumann problem and Stein neighborhood bases*, Ph.D. thesis, Texas A&M University, TX, 2006, available at `http://handle.tamu.edu/1969.1/3879`.

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